Comment on: Diffusion through a slab

U. D. J. Gieseler & J. G. Kirk
Max-Planck-Institut für Kernphysik, Postfach 10 39 80, D-69029 Heidelberg, Germany
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Mahan [J. Math. Phys. **36**, 6758 (1995)] has calculated the transmission coefficient and angular distribution of particles which enter a *thick* slab at *normal* incidence and which diffuse in the slab with linear anisotropic, non-absorbing, scattering. Using orthogonality relations derived by McCormick & Kuščer [J. Math. Phys. **6**, 1939 (1965); **7**, 2036 (1966)] for the eigenfunctions of the problem, this calculation is generalised to a boundary condition with particle input at *arbitrary* angles. It is also shown how to use the orthogonality relations to relax in a simple way the restriction to a thick slab.

We consider the equation of radiative transfer with anisotropic scattering in a uniform slab, which occupies the space $0 < z < D$, together with a boundary condition which allows particles to enter the slab through the surface $z = 0$ at an angle $\theta = \arccos\mu_0$ to the normal:[1]

$$\mu\frac{\partial}{\partial z}f(z,\mu) + f(z,\mu) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu' f(z,\mu') + \frac{3}{2}\,\mu\, g_1 \int_{-1}^{1} \mathrm{d}\mu'\mu' f(z,\mu')\,, \tag{1}$$

$$\left.\begin{array}{lcl} f(0,\mu) & = & 2\,\delta(\mu-\mu_0) \\ f(D,-\mu) & = & 0 \end{array}\right\} \text{ for } \mu > 0\,. \tag{2}$$

For thick slabs ($D \gg 1$), Mahan[2] has presented a solution to this problem which is valid only for $\mu_0 = 1$. Generalisation to arbitrary μ_0 is of interest when, for example, the particles which enter the slab come from a point source at finite distance, or diffuse before entering the slab. These problems require an integration over the range of incident angles. Even for collimated beams, the experimental situation is generally one in which the particles are not normally incident. Mahan's method is not readily generalised to solve this problem: his Eq. (78) does not hold when $\mu_0 \neq 1$, since then $A^{-1}(\mu_0) \neq 0$.

The general solution to Eq. (1) is[3]

$$\begin{aligned} f(z,\mu) = a_s + 3j\left[\mu - z(1-g_1)\right] + &\int_0^1 \mathrm{d}\nu \left\{\frac{M_L(\nu)}{\nu-\mu}e^{-z/\nu} + \delta(\nu-\mu)A(\nu)M_L(\nu)e^{-z/\nu}\right\} \\ + &\int_{-1}^0 \mathrm{d}\nu \left\{\frac{M_R(\nu)}{\nu-\mu}e^{(D-z)/\nu} + \delta(\nu-\mu)A(\nu)M_R(\nu)e^{(D-z)/\nu}\right\}\,, \end{aligned} \tag{3}$$

where the constants a_s and j, and the functions $M_L(\nu)$ and $M_R(\nu)$ are to be determined from the boundary conditions. The explicit form of the function $A(\mu)$ reads[4]

$$A(\mu) = -2\,\frac{Q_1(\mu)}{P_1(\mu)} = \frac{2}{\mu}(1-\mu\,\mathrm{arctanh}\,\mu) = \frac{2}{\mu}\,\lambda(\mu)\,, \tag{4}$$

where $\lambda(\mu)$ is defined by McCormick & Kuščer[5]. To apply the orthogonality relations, it is necessary to rewrite the solution in terms of the eigenfunctions used by McCormick & Kuščer[6]:

$$\phi_\nu(\mu) = \frac{\nu}{2}\,\mathrm{P}\,\frac{1}{\nu-\mu} + \lambda(\nu)\delta(\nu-\mu)\,, \tag{5}$$

which have the property

$$\phi_{-\nu}(\mu) = \phi_\nu(-\mu)\,. \tag{6}$$

Equation (3) can then be written

$$f(z,\mu) = a_s + 3j\left[\mu - z(1-g_1)\right] + \int_0^1 \mathrm{d}\nu \tilde{M}_L(\nu)\phi_\nu(\mu)e^{-z/\nu} + \int_0^1 \mathrm{d}\nu \tilde{M}_R(-\nu)\phi_{-\nu}(\mu)e^{(z-D)/\nu}\,, \tag{7}$$

where we have absorbed the factor $2/\mu$ into the definition of the functions $\tilde{M}_R(\mu)$ and $\tilde{M}_L(\mu)$ according to

$$\tilde{M}_R(\mu) := \frac{2}{\mu}\,M_R(\mu)\,, \qquad \tilde{M}_L(\mu) := \frac{2}{\mu}\,M_L(\mu)\,. \tag{8}$$

The boundary conditions [Eq. (2)] then become

$$2\delta(\mu-\mu_0) = a_s + 3j\mu + \int_0^1 \mathrm{d}\nu \tilde{M}_L(\nu)\phi_\nu(\mu) + \int_0^1 \mathrm{d}\nu \tilde{M}_R(-\nu)\phi_{-\nu}(\mu)e^{-D/\nu}\,, \tag{9}$$

$$0 = a_s - 3j\mu - 3jD(1-g_1) + \int_0^1 \mathrm{d}\nu \tilde{M}_L(\nu)\phi_{-\nu}(\mu)e^{-D/\nu} + \int_0^1 \mathrm{d}\nu \tilde{M}_R(-\nu)\phi_\nu(\mu)\,. \tag{10}$$

Defining

$$B_\pm(\nu) := \frac{1}{2}\left[\tilde{M}_L(\nu) \pm \tilde{M}_R(-\nu)\right] \tag{11}$$

and adding and subtracting Eqs. (9) and (10) leads to:

$$\delta(\mu-\mu_0) = \left\{ \begin{array}{c} a_s \\ 3j\mu \end{array} \right\} \mp \frac{3}{2}jD(1-g_1) + \int_0^1 B_\pm(\nu)\phi_\nu(\mu)\,\mathrm{d}\nu \pm \int_0^1 B_\pm(\nu)e^{-D/\nu}\phi_{-\nu}(\mu)\,\mathrm{d}\nu\,. \tag{12}$$

In order apply the orthogonality relations, these equations must be multiplied by a weight function. This function, denoted here and in McCormick & Kuščer[7] by $\gamma(\mu)$, is related, but not identical, to the $\gamma(\mu)$ defined by Mahan[2], and is given by[8]

$$\gamma(\mu) = \frac{3}{2}\,\frac{\mu}{X(-\mu)}\,; \qquad 0 \le \mu \le 1\,. \tag{13}$$

The function $X(-\mu)$ can be written in terms of the Ambartsumian function[9] $\psi(\mu)$ or the Chandrasekhar H-function[10]. In the limit $c \to 1$ these relationships are[11,12]

$$X(-\mu) = \frac{\sqrt{3}}{\psi(\mu)} = \frac{\sqrt{3}}{H(\mu)}\,. \tag{14}$$

Tables of $X(-\mu)$, for $0 \le \mu \le 1$ are given by Case & Zweifel[11]; numerical evaluation is straightforward using the representation[13]

$$X(-\mu) = \exp\left\{ \frac{-c}{2}\int_0^1 \mathrm{d}x \left(1 + \frac{c\,x^2}{1-x^2}\right) \frac{\ln(x+\mu)}{[1 - c\,x\,\mathrm{arctanh}(x)]^2 + (\pi c\,x/2)^2} \right\}, \tag{15}$$

where c is the albedo for single scattering, equal to unity in the case discussed here. We now multiply Eq. (12) by $\gamma(\mu)$ and integrate over μ from 0 to 1. The integrals over μ can be solved using relations provided by McCormick & Kuščer[7] (the numbers above the equals signs in the following refer to the relevant equation numbers):

$$\int_0^1 \gamma(\mu)\,\mathrm{d}\mu \overset{16}{=} \gamma_0 \overset{63}{=} 1\,, \tag{16}$$

$$\int_0^1 \gamma(\mu)\mu\,\mathrm{d}\mu \overset{16}{=} \gamma_1 \overset{25}{=} \bar{\nu}\gamma_0 \overset{63}{=} \bar{\nu} \overset{83}{=} z_0\big|_{b=0} = 0.7104\,, \tag{17}$$

$$\int_0^1 \phi_\nu(\mu)\gamma(\mu)\,\mathrm{d}\mu \stackrel{69}{=} 0\,, \tag{18}$$

$$\int_0^1 \phi_{-\nu}(\mu)\gamma(\mu)\,\mathrm{d}\mu \stackrel{70}{=} \frac{3}{4}\,\frac{\nu^2}{\gamma(\nu)} \;=\; \frac{\nu}{2}\,X(-\nu)\,. \tag{19}$$

If we denote the extrapolation distance for the Milne problem in the case of isotropic scattering $z_0\big|_{b=0} = 0.7104$ by simply z_0, then, using the above relations, Eq. (12) becomes

$$\frac{3}{2}\,\frac{\mu_0}{X(-\mu_0)} = \left\{ \begin{array}{c} a_s \\ 3jz_0 \end{array} \right\} \mp \frac{3}{2} jD(1-g_1) \pm \int_0^1 B_\pm(\nu) e^{-D/\nu}\,\frac{\nu}{2}\,X(-\nu)\,\mathrm{d}\nu\,. \tag{20}$$

The functions $B_\pm(\mu)$ can be calculated by multiplying Eq. (12) by $\phi_{\nu'}(\mu)\gamma(\mu)$ and integrating over μ from 0 to 1. Using the orthogonality relations[14], one finds inhomogeneous Fredholm equations for $B_\pm(\mu)$ which can be solved by Neumann iteration[15]. In the thick slab approximation, where terms of order e^{-D} are ignored, these Fredholm equations are trivially solved. Equation (20) for a_s and j is then also trivial and independent of $B_\pm(\mu)$:

$$\frac{3}{2}\,\frac{\mu_0}{X(-\mu_0)} = \left\{ \begin{array}{c} a_s \\ 3jz_0 \end{array} \right\} \mp \frac{3}{2} jD(1-g_1)\,. \tag{21}$$

Once the functions $B_\pm(\mu)$, and hence $M_L(\mu)$ and $M_R(\mu)$ have been found, Eq. (21) provides a_s and j and, therefore, the density $f(\mu, z)$. It is in principle possible to follow this procedure taking into account higher order terms $\propto e^{-D}$. However, the equations become complicated in this case.

Equations (21) enable the transmission coefficient T to be evaluated directly. In terms of the X function we find:

$$T = j = \frac{\mu_0}{X(-\mu_0)}\,\frac{1}{D(1-g_1) + 2z_0}\,. \tag{22}$$

This result generalises to arbitrary μ_0 $(0 \le \mu_0 \le 1)$ the result of Mahan[2] [Eq. (110)], with which it agrees for $\mu_0 = 1$. In the case of isotropic scattering, $g_1 = 0$, Eq. (22) is in agreement with the result of McCormick & Mendelson[12] [Eq. (35)].

Finally, it should be noted that McCormick & Kuščer[16] have also found orthogonality relations which can be used to solve half-space transport problems with higher order anisotropy.

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[1] Reference 2, Eqs. (4) - (6).
[2] G. D. Mahan, J. Math. Phys. **36**, 6758 (1995).
[3] Reference 2, Eqs. (88) and (89).
[4] Reference 2, Eq. (29).
[5] Reference 7, Eq. (5) in which, for the case of pure scattering considered here, the limit $c \to 1$ must be taken.
[6] Reference 7, Eq. (4), and Reference 11, Section 6.9.
[7] N. J. McCormick, I. Kuščer, J. Math. Phys. **6**, 1939 (1965).
[8] Reference 7, Eq. (15), taking the limit $c \to 1$.
[9] Reference 2, Eq. (7).
[10] S. Chandrasekhar, *Radiative Transfer*, (Dover P, New York, 1960), using the definition of H appropriate for isotropic scattering.
[11] K. M. Case, P. F. Zweifel, *Linear Transport Theory*, (Addison-Wesley, London, 1967).
[12] N. J. McCormick, M. R. Mendelson, Nucl. Sci. Eng. **20**, 462 (1964).
[13] Reference 11, p 130, Eq. (39).
[14] Reference 7, Eqs. (64) and (65).
[15] Reference 12 treats the case of isotropic scattering.
[16] N. J. McCormick, I. Kuščer, J. Math. Phys. **7**, 2036 (1966).