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MONTE CARLO SIMULATION OF PARTICLE TRANSPORT IN BRAIDED MAGNETIC FIELDS

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ABSTRACT

We present Monte Carlo simulations of particle transport in wandering magnetic field lines. When particles undergo diffusion, due to small scale inhomogeneities, and are tied to a braided field line which also diffuses, we reproduce the sub-diffusive transport regime predicted by theory. Extending the simulation to particles undergoing pitch-angle scattering, and using a random walk approximation to diffusion for the field lines, makes it possible to identify those points at which a shock front is crossed, a prerequisite for the treatment of acceleration. We present detailed comparison of the numerically simulated propagator and of the density profile which results from continuous injection at a moving boundary with analytic results.

INTRODUCTION

Despite the fact that analytic solutions are often tractable, the problem of diffusive particle acceleration at a shock front, has frequently been the subject of numerical simulation using the Monte-Carlo method. This has proved particularly useful in cases where boundary conditions in momentum or configuration space are imposed, (for a review see Jones & Ellison 1991), or in where the transport is not strictly diffusive (Kirk & Schneider 1987, Naito & Takahara 1995) or both (Ellison et al. 1997).

Recently, (Dendy et al. 1995, Duffy et al. 1995, Kirk et al. 1996) an analytic theory of particle acceleration at shocks has been developed which assumes anomalous transport instead of diffusion, and which is applicable to shock fronts traversing a braided or stochastic magnetic field. A Monte-Carlo simulation of this acceleration process must face the problem that anomalous transport is inherently non-Markovian, so that the simple conceptual approach of a series of independent stochastic events is insufficient. In this paper we present two techniques we have developed to overcome this problem. Each enables a Monte-Carlo simulation of '$\alpha = 1/2$ sub-diffusion' (Ragot & Kirk 1997). This can be visualised as the diffusion of particles along a magnetic field line which itself diffuses in the plane normal to its average direction.

DOUBLE DIFFUSION

The first method is based on the fact that it is possible to solve the diffusion problem exactly for the propagator in an infinite homogeneous medium. Consider a magnetic field which is predominantly along the z direction, but contains a small stochastic component in the plane perpendicular to this direction, which causes the field lines to diffuse. Given that the field line passes through the point $x = x_0$, $z = z_0$, the probability of finding it displaced at z is

$$P(x,z) = \frac{1}{2\sqrt{\pi D_{\rm M}|z-z_0|}} \exp\left[-(x-x_0)^2/4D_{\rm M}|z-z_0|\right] \qquad (1)$$

where $D_{\rm M}$ is the diffusivity of the magnetic field. If we now allow a particle to diffuse along the field, starting at the point (x_0, z_0) at time t, the probability of finding it at z at time $t + \Delta t$ is

$$P(z, \Delta t) = \frac{1}{2\sqrt{\pi\kappa_{\|}\Delta t}} \exp\left[-(z - z_0)^2/4\kappa_{\|}\Delta t\right] \quad (2)$$

where $\kappa_{\|}$ is the diffusion coefficient along the field. The first step in the Monte-Carlo simulation is, therefore, to select z from the distribution (2) given Δt. Denoting the result by z_1, one then inserts $z = z_1$ into (1) and selects a value of x, denoted by x_1.

In order to implement sub-diffusive transport, each point visited on the field line i.e., (x_0, z_0) and (x_1, z_1) is stored. The next step is again to choose a value z_2 from Eq. (2). There are then three possibilities: $z_2 < \min(z_0, z_1)$, or $z_2 > \max(z_0, z_1)$ or z_2 lies between the two known locations of the field line. In order to find the new $x = x_2$ corresponding to z_2, we compute the conditional probability that the field line goes through the point (x, z_2), given that it also passes through (x_0, z_0), (x_1, z_1). In the first two cases, this is found from Eq. (1) by simply replacing $|z - z_0|$ by $z_2 - \max/\min(z_0, z_1)$ and $x - x_0$ by the corresponding quantity. If, on the other hand, z_2 lies between two known points on the field line, the conditional probability can be computed as

$$P(x, z) = \frac{1}{2\sqrt{\pi D_{\mathrm{M}}|\bar{z}|}} \exp\left[-(x - \bar{x})^2/4D_{\mathrm{M}}|\bar{z}|\right] \quad (3)$$

where $\bar{z} = (z_1 - z)(z - z_0)/(z_1 - z_0)$ and $\bar{x} = [x_0(z_1 - z) + x_1(z - z_0)]/(z_1 - z_0)$ (e.g., Feynman & Hibbs 1965). The new point with x_2 chosen from this distribution, is added to the vector containing the field line points in such a way that the elements are ordered in increasing values of z_i.

This technique works accurately and quickly, but has two disadvantages: (1) the total number of simulation steps is limited, since the storage space occupied by the array describing the magnetic field grows continuously, and (2) for the purpose of simulating accelerating it is unsuitable, since the number of times a particle crosses a given surface (e.g., a shock front) is not accessible. (A naive count yields a fractal set: $N_{\mathrm{cross}} \propto \Delta t^{-1/2}$ e.g., Mandelbrot 1982.)

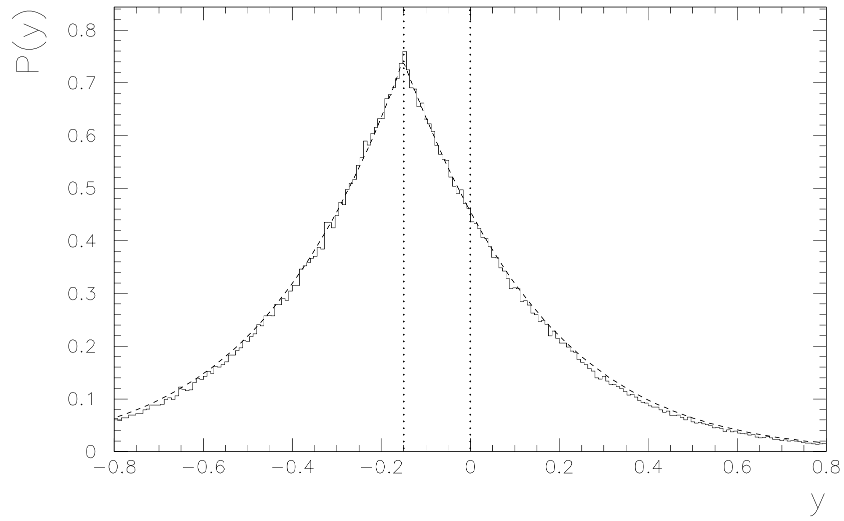

Fig. 1: The propagator for sub-diffusive transport from a Monte-Carlo simulation at dimensionless time $\tau = 0.15$ (see Eq. 5) using a random walk for the magnetic field line and pitch angle scattering for the particle's transport along the field line. The analytic propagator is the dashed line. The plasma into which the field lines are frozen moves with uniform speed in the negative y direction. Injection took place at the point $y = 0$, at dimensionless time $\tau = 0$.

RANDOM WALKS

These problems can be overcome by simulating diffusion approximately using a simple random walk. The only subtlety is that in order for a particle to sub-diffuse, it must perform two random walks simultaneously – one corresponding to the field line, and one along the field line. Whilst doing the second, the entire history of the first must be known. It is possible to achieve this using a random number generator such as is used for encryption (Press et al. 1986). Here, a random sequence is 'labeled' by a single integer – once this integer is given, any member of the sequence is immediately recalled by specifying its position in the sequence. In our application, this means that each individual random field line is specified at a large (essentially infinite) number of grid points. The x values corresponding to these points are not stored, but can be recalled because they correspond to a labeled sequence of random numbers. In between the grid points, we assume the field to be uniform, so that at any value of z selected, for example, from Eq. (2), a value of x could be found by linear interpolation.

Diffusion along the field line can, or course, be simulated in a similar manner, by selecting a mean free path and specifying an unperturbed trajectory between scattering events. We have used this technique to simulate a particle whose propagation can be described using the guiding centre approximation, together with a small random change in pitch angle at each scattering. In this way, the trajectory of a particle is specified at each point, except for the unknown gyro phase. On crossing a shock front, it is then possible to use the conservation of the first adiabatic invariant to determine the effect on the trajectory (Kirk & Heavens 1989).

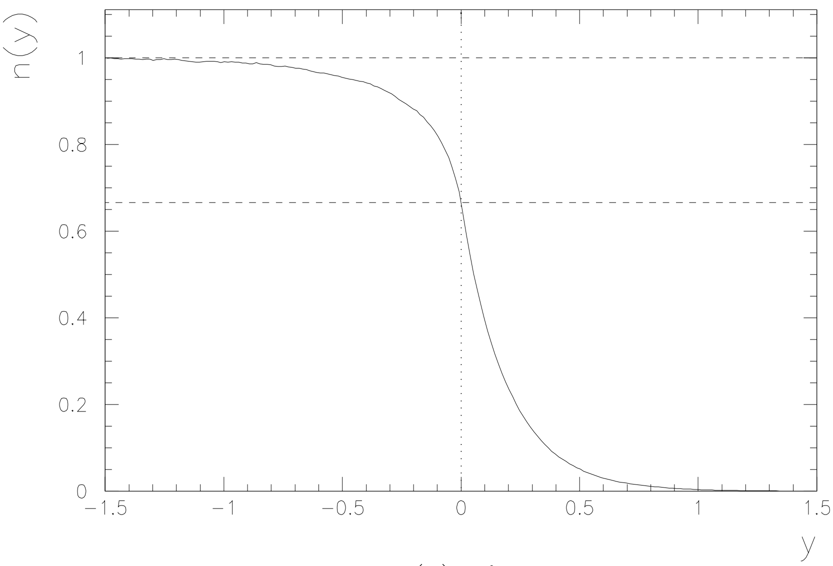

Fig. 2: The stationary density $n(y)$ of particles injected continuously at the point $y = 0$ into plasma moving at uniform speed in the negative y direction. As in Figure 1, sub-diffusive transport is simulated using the random walk method for the field and pitch angle scattering for the particles. The upper dashed line gives the asymptotic value at large negative y. The analytically predicted value of the density at the point of injection is shown by the lower dashed line, which is $2/3$ times that at $y = -\infty$. The simulation reproduces this result accurately.

We have tested this algorithm by computing the propagator of particles injected at a moving boundary (such as a shock front) into a uniform medium. Figure 1 shows the distribution of particle positions at fixed time after injection, compared with the analytic result (Rax & White 1992; Duffy et al. 1995; Kirk et al. 1996):

$$P(\xi,\tau) = \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}s}{\sqrt{s\tau}} \exp\left[-\xi^2/(4s) - s^2/(4\tau)\right] \tag{4}$$

where

$$\xi = \frac{u^{1/3}}{D_{\mathrm{M}}^{2/3}\kappa_\parallel^{1/3}}x\,, \qquad \tau = \frac{u^{4/3}}{D_{\mathrm{M}}^{2/3}\kappa_\parallel^{1/3}}t \tag{5}$$

with u the speed of the boundary. In this figure, we have plotted P as a function of y, the dimensionless distance from the moving boundary $y = \xi - \tau$, at time $\tau = 0.15$ after injection, chosen such that the plasma moves a distance indicated by the dotted lines, to the left. The analytic formula for $\alpha = 1/2$ sub-diffusion (Eq. 4) is shown by the dashed line.

As a further check, we have simulated the stationary density profile expected when particles are injected continuously at the moving boundary. This profile and, especially, its intercept at the boundary $y = 0$, plays an important role in the analytic theory of particle acceleration in the presence of anomalous transport. Figure 2 shows the result. Analytically, one expects the density at the boundary to be 2/3 the value far downstream. This is indicated by dashed lines in the figure. An analogous test of the double-diffusion method is presented in Kirk et al. (1997).

CONCLUSIONS

The two Monte-Carlo simulation techniques presented have been tested against the analytic propagator. Each is rapid and accurate for large numbers of time-steps. The second method, which departs from the analytic propagator for a small number of steps, has the advantage that it contains a physically reasonable particle trajectory, which is essential for the simulation of acceleration at shocks.

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