Appendix G

Solutions to Problems in Chapter 7

G.1 Problem 7.1

The maximum radiated power is

\[ EIRP = GP = 100 \text{mW} = 20 \text{dBm} \]  \hspace{1cm} \text{(G.1)}

First, we determine the transmit power of the initial configuration (line loss \( a_1 = 2.5 \text{dB} \) and antenna gain \( G_1 = 5 \text{dBi} \)).

\[ P_{TX1} = EIRP - G_1 + a_1 = 20 \text{dBm} - 5 \text{dBi} + 2.5 \text{dB} = 17.5 \text{dBm} \]  \hspace{1cm} \text{(G.2)}

For the modified configuration (line loss \( a_2 = 1 \text{dB} \) and antenna gain \( G_2 = 15 \text{dBi} \)) the transmit power is

\[ P_{TX2} = EIRP - G_2 + a_2 = 20 \text{dBm} - 15 \text{dBi} + 1 \text{dB} = 6 \text{dBm} \]  \hspace{1cm} \text{(G.3)}

When the second configuration acts as a receiver the larger antenna gain and reduced line loss lead to increased receive power and increased range.

G.2 Problem 7.2

We will simulate three antenna structures with a commercial EM simulation program (EMPIRE from IMST).

- Monopole antenna
- Top loaded monopole antenna
- Inverted-F antenna

The operational frequency shall be 2.45 GHz.

Monopole antenna

Figure G.1 shows the monopole over conducting ground. The geometrical length (\( \ell = 2.8 \text{cm} \)) is slightly smaller than a quarter wavelength (\( \lambda/4 = 3.06 \text{cm} \)). Figure G.2 shows the reflection coefficient for a port reference impedance of \( Z_0 = 50 \Omega \). At 2.45 GHz we observe low reflection.
The input impedance $Z_{\text{in}}$ is given in Figure G.3. At a frequency of $2.45\,\text{GHz}$ the imaginary part is close to zero and the real part is approximately $36\,\Omega$. Figure G.4 shows the radiation pattern of the monopole antenna. The simulated directivity of $D = 5.19\,\text{dBi}$ is close to the theoretical value of $5.15\,\text{dBi}$. 
Figure G.3: Real (green line) and imaginary (blue line) parts of input impedance (monopole antenna).

Figure G.4: Radiation pattern of monopole antenna.

**Top loaded monopole antenna**

Figure G.5 shows a top loaded monopole antenna. The cylindrical structure at the top of the monopole has a diameter of 12 mm. The overall height of the antenna is \( h = 16 \text{ mm} \). The height is significantly reduced compared to the initial length \( \ell \) of the quarterwave monopole antenna.
\( h = 16 \text{ mm} < \ell = 28 \text{ mm} \approx \lambda/4 \). Figure G.6 shows the reflection coefficient of the top loaded monopole antenna. Due to a reduced real part of the input impedance the matching is poor compared to the initial quarterwave monopole (see Figure G.7).

The radiation pattern in Figure G.8 shows only minor changes compared to the initial monopole antenna.
Inverted-F antenna

Figure G.9 shows an inverted-F antenna over conducting ground. The height of $h_F = 14\, \text{mm}$ is only half of the initial monopole length ($h_F = \ell/2$). The feedpoint can be adjusted to achieve a good matching to $50\, \Omega$ as shown in Figure G.10. The frequency dependence of the input
impedance given in Figure G.11 is quite different from the input impedance of a monopole. For low frequencies a monopole shows capacitive behaviour (negative imaginary part) whereas an inverted-F antenna shows inductive behaviour (positive imaginary part). As opposed to a
Figure G.11: Real (green line) and imaginary (blue line) parts of input impedance (inverted-F antenna)

A monopole antenna an inverted-F antenna shows radiation also in vertical direction (see Figure G.12).

Figure G.12: Radiation pattern of inverted-F antenna
G.3 Problem 7.3

We use Equation 7.49 and 7.50 (book page 273) to estimate the patch length $L$. Since the equations cannot be solved directly for $L$, we try values numerically and get a length of $L = 19.6$ mm. Using the given relation of $W = 1.5L$ the width is $W = 29.4$ mm.

The feedpoint location is given by Equation 7.52 and 7.53 (book page 275). We get $x_f = 5.6$ mm and $y_f = 14.7$ mm. Figure G.13 shows the patch (red) over a substrate (grey) as well as the feedpoint location (light red). Furthermore the radiation pattern is given. The directivity is $D = 6.44$ dBi. (The simulations have been performed with the EM simulation software EMPIRE from IMST.)

![Image of radiation pattern and geometry of the designed patch antenna](image)

Figure G.13: Radiation pattern and geometry of the designed patch antenna.

The reflection coefficient is shown in Figure G.14. The initial design shows matching at a frequency of 3.89 GHz (about 3% lower than the specified frequency of 4 GHz). In order to achieve matching at the specified frequency of 4 GHz we reduce the length of the patch slightly from 19.6 mm to 19.0 mm and change the feedpoint location from $f_1 = 5.6$ mm to $f_1 = 5.4$ mm. The resulting reflection coefficient is shown in Figure G.15. The initial design provided a good starting point for the subsequent optimization.
Figure G.14: Reflection coefficient of the designed patch antenna (initial design).

Figure G.15: Reflection coefficient of the patch antenna (initial design (red line) and optimized design (black line)).
G.4 Problem 7.4

The magnetic vector potential \( \vec{A} \) (see Equation (7.29) on book page 263) is given as

\[
\vec{A} = \frac{\mu_0 I \ell}{4\pi} \frac{e^{-jkr}}{r} \hat{e}_z \quad \text{(G.4)}
\]

In Equation G.4 we use mixed Cartesian and spherical coordinates. In order to perform the calculation in spherical coordinates we express the unit vector \( \hat{e}_z \) in spherical coordinates (see Equation (A.14) on book page 324).

\[
\hat{e}_z = \vec{e}_r \cos \vartheta - \vec{e}_\vartheta \sin \vartheta \quad \text{(G.5)}
\]

Hence, no \( \varphi \)-component exists (\( A_\varphi = 0 \)).

We determine the magnetic field strength \( \vec{H} \) using Equation (7.23).

\[
\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} = \frac{I \ell}{4\pi} \nabla \times \left( \frac{e^{-jkr}}{r} \left[ \vec{e}_r \cos \vartheta - \vec{e}_\vartheta \sin \vartheta \right] \right) \quad \text{(G.6)}
\]

The curl operator in spherical coordinates reads

\[
\nabla \times \vec{A} = \frac{1}{r \sin \vartheta} \left( \frac{\partial}{\partial \vartheta} \left( r^2 A_r \sin \vartheta \right) \right) \hat{e}_r + \frac{1}{r} \left( \frac{\partial A_\vartheta}{\partial \varphi} - \frac{\partial (r A_\varphi)}{\partial r} \right) \hat{e}_\varphi + \frac{1}{r \sin \vartheta} \left( \frac{\partial}{\partial \varphi} \left( r A_\varphi \sin \vartheta \right) \right) \hat{e}_\varphi \quad \text{(G.7)}
\]

Since there is no \( \varphi \)-component (\( A_\varphi = 0 \)) and the remaining components are no functions of \( \varphi \) (\( A_r; A_\theta \neq \text{fct}(\varphi) \)) the expression is simplified to

\[
\vec{H} = \frac{I \ell}{4\pi r} \left( \frac{\partial}{\partial \vartheta} \left( r \left( -\frac{e^{-jkr}}{r} \right) \sin \vartheta \right) \right) \hat{e}_\varphi - \frac{I \ell}{4\pi r} \frac{\partial}{\partial \vartheta} \left( \frac{e^{-jkr}}{r} \cos \vartheta \right) \hat{e}_\varphi \quad \text{(G.8)}
\]

So we end up with the following result (see Equation (7.34)).

\[
\vec{H} = \frac{I \ell}{4\pi r^2} \left( 1 + jkr \right) \sin \vartheta \hat{e}_\varphi \quad \text{(G.9)}
\]

The electrical field strength \( \vec{E} \) is given by Equation (7.24) (see book page 263).

\[
\vec{E} = \frac{\nabla (\nabla \cdot \vec{A})}{j\omega \mu_0} - j\omega \vec{A} \quad \text{(G.10)}
\]

The divergence operator in spherical coordinates reads:

\[
\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 A_r \right) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( A_\vartheta \sin \vartheta \right) + \frac{1}{r \sin \vartheta} \frac{\partial A_\varphi}{\partial \varphi} \quad \text{(G.11)}
\]
We get
\[
\nabla \cdot \vec{A} = \frac{\mu_0 I \ell}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{e^{-jkr}}{r} \cos \vartheta \right) + \frac{\mu_0 I \ell}{4\pi} \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \frac{e^{-jkr}}{r} (-\sin^2 \vartheta) \right)
\]
\[= \frac{\mu_0 I \ell}{4\pi} \left[ \cos \vartheta \frac{1}{r^2} \left( r(-jk)e^{-jkr} + e^{-jkr} \right) + \frac{1}{r \sin \vartheta} \frac{e^{-jkr}}{r} (-2 \sin \vartheta \cos \vartheta) \right]
\]
\[\text{(G.12)}
\]
Combining all terms yields
\[
\nabla \cdot \vec{A} = \frac{\mu_0 I \ell}{4\pi} \frac{e^{-jkr}}{r^2} (1 + jkr) \cos \vartheta
\]
\[\text{(G.13)}
\]
Now we apply the gradient operator which is given as
\[
\nabla \phi = \frac{\partial \phi}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \vartheta} \vec{e}_\vartheta + \frac{1}{r \sin \vartheta} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi
\]
where \( \phi = \nabla \cdot \vec{A} \)
\[\text{(G.15)}
\]
Since \( \nabla \cdot \vec{A} \) is no function of \( \varphi \) the last term is zero. So we consider first the derivative with respect to \( r \):
\[
\frac{\partial (\nabla \cdot \vec{A})}{\partial r} = -\frac{\mu_0 I \ell \cos \vartheta}{4\pi} \frac{\partial}{\partial r} \left[ \frac{e^{-jkr}}{r^2} + jk \frac{e^{-jkr}}{r} \right]
\]
\[\text{(G.16)}
\]
The quotient rule
\[
\left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2}
\]
\[\text{(G.17)}
\]
gives us
\[
\frac{\partial (\nabla \cdot \vec{A})}{\partial r} = -\frac{\mu_0 I \ell \cos \vartheta}{4\pi} \left[ -jk e^{-jkr} r^2 - e^{-jkr} 2r + jk (-jk) e^{-jkr} r - e^{-jkr} r \right]
\]
\[\text{(G.18)}
\]
\[\frac{\partial (\nabla \cdot \vec{A})}{\partial r} = -\frac{\mu_0 I \ell \cos \vartheta}{4\pi} \left[ 2jk \frac{e^{-jkr}}{r^2} - 2 \frac{e^{-jkr}}{r^3} + k^2 \frac{e^{-jkr}}{r} \right]
\]
\[\text{(G.19)}
\]
Hence, the radial component of the electric field strength yields
\[
E_r = \frac{1}{j \omega \mu_0 \varepsilon_0} \left( -\frac{\mu_0 I \ell}{4\pi} \cos \vartheta \left[ -2jk \frac{e^{-jkr}}{r^2} - 2 \frac{e^{-jkr}}{r^3} + k^2 \frac{e^{-jkr}}{r} \right] \right)
\]
\[= \frac{\mu_0 I \ell}{4\pi} \cos \vartheta \left[ \frac{2k}{\omega \mu_0 \varepsilon_0 r^3} - \frac{2}{j \omega \mu_0 \varepsilon_0 r^3} \right]
\]
\[\text{(G.20)}
\]
With
\[k^2 = \omega^2 \varepsilon_0 \mu_0
\]
we get
\[
E_r = \frac{\mu_0 I \ell}{4\pi} \cos \vartheta e^{-jkr} \left[ \frac{2k}{\omega \mu_0 \varepsilon_0 r^3} + \frac{2}{j \omega \mu_0 \varepsilon_0 r^3} \right]
\]
\[\text{(G.22)}
\]
Our final result for the radial component of the electrical field strength is

\[ E_r = \frac{I \ell}{j 2\pi \omega \varepsilon_0} \cos \vartheta \frac{e^{-jkr}}{r^3} (1 + jkr) \]  

(G.23)

Finally, we consider the \( \vartheta \)-component of the electrical field strength.

\[ E_\vartheta = \frac{1}{j \omega \mu_0 \varepsilon_0} \frac{1}{r} \frac{\partial (\nabla \cdot \vec{A})}{\partial \vartheta} - j \omega A_\vartheta \]  

(G.24)

\[ = \frac{1}{j \omega \mu_0 \varepsilon_0} \frac{1}{r} \left( -\frac{\mu_0 I \ell}{4\pi} \cdot \frac{e^{-jkr}}{r^2} (1 + jkr) (-\sin \vartheta) \right) - j \omega \left( -\frac{\mu_0 I \ell}{4\pi} \cdot \frac{e^{-jkr}}{r} \sin \vartheta \right) \]  

(G.25)

With

\[ k^2 = \omega^2 \varepsilon_0 \mu_0 \]  

\( \rightarrow \)  

\[ \omega = \frac{k^2}{\omega \varepsilon_0 \mu_0} \]  

(G.26)

we get

\[ E_\vartheta = \frac{I \ell}{j 4\pi \omega \varepsilon_0} \frac{e^{-jkr}}{r^3} \sin \vartheta (1 + jkr - (kr)^2) \]  

(G.27)

G.5 Problem 7.5

The elements of a two-dimensional array antenna are located in \( xy \)-plane (\( z = 0 \)). Figure 7.28 (book page 288) shows the arrangement. The lower left element is positioned at \( x = y = 0 \), the corresponding indexes are \( m = n = 0 \).

Exciting all elements with equal phase results in a main beam direction perpendicular to the antenna plane, i.e. the main lobe points into \( z \)-direction (\( \vartheta = 0^\circ \)). By changing the phase the main lobe may be tilted into another direction. The direction may be described either by the angles \( \varphi_0 \) and \( \vartheta_0 \) or by the normal vector \( \vec{n} \).

\[ \vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \cos \varphi_0 \sin \vartheta_0 \\ \sin \varphi_0 \sin \vartheta_0 \\ \cos \varphi_0 \end{pmatrix} \]  

(G.28)

A plane wave that travels in the direction of the normal vector has planes of equal phase that are perpendicular to the direction of propagation. Let us consider such a plane that includes the origin of the coordinate system.

\[ n_x x + n_y y + n_z z = 0 \]  

(G.29)

In order to produce constructive superposition in the direction of the normal vector the phase (or delay time) of each individual antenna element has to be adjusted. The antenna elements are located at the following points in space.

\[ P = \begin{pmatrix} m d_x \\ n d_y \\ 0 \end{pmatrix} \]  

(G.30)
The distance \(d_{mn}\) between antenna element and plane of equal phase is given by (Hesse normal form)

\[
d_{mn} = \frac{n_xx + n_yy + n_zz - 0}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = md_x \cos \varphi_0 \sin \theta_0 + nd_y \sin \varphi_0 \sin \theta_0
\]

(G.31)

Due to the speed of propagation \(c_0\) we determine the delay times as

\[
\Delta t_{mn} = \frac{d_{mn}}{c_0}
\]

(G.32)

For a given frequency \(f_0 = c_0/\lambda_0\) the phases are

\[
\Delta \Phi_{mn} = \Delta t_{mn} \frac{c_0 \cdot 360^\circ}{\lambda_0} = \frac{d_{mn}}{c_0} \cdot \frac{c_0 \cdot 360^\circ}{\lambda_0}
\]

\[
= \frac{360^\circ}{\lambda_0} [md_x \cos \varphi_0 \sin \theta_0 + nd_y \sin \varphi_0 \sin \theta_0]
\]

(G.33)

(G.34)

Remember: A negative phase represents a delay in the time domain.

(Last modified: 14.02.2013)